

Improved nonuniform Berry–Esseen-type bounds

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Abstract: New nonuniform Berry–Esseen-type bounds for sums of independent random variables are obtained, motivated by recent studies concerning such bounds for nonlinear statistics. The proofs are based on the Chen–Shao concentration techniques within the framework of the Stein method.

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1. Summary and discussion

For any natural n , let ξ_1, \dots, ξ_n be independent zero-mean random variables (r.v.'s) such that

$$\mathbb{E} \xi_1^2 + \dots + \mathbb{E} \xi_n^2 = 1, \quad (1.1)$$

and let

$$S := \xi_1 + \dots + \xi_n.$$

Take any $v \in (0, \infty)$ and introduce

$$\beta_v := \mathbb{E} g(\xi_1/v) + \dots + \mathbb{E} g(\xi_n/v),$$

where

$$g(x) := x^2 \wedge |x|^3 \quad (1.2)$$

for all $x \in \mathbb{R}$. Observe that for any $p \in [2, 3]$ one has $g(x) \leq |x|^p$ for all real x and hence

$$\beta_v \leq \mu_p/v^p,$$

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where

$$\mu_p := \mathbf{E} |\xi_1|^p + \cdots + \mathbf{E} |\xi_n|^p \quad (1.3)$$

for all $p \in (0, \infty)$.

Take also any $w \in (0, \infty)$ and let $\xi_{w,1}, \dots, \xi_{w,n}$ be any independent r.v.'s such that for each $i = 1, \dots, n$

$$\xi_{w,i} = \xi_i \text{ on the event } \{\xi_i \leq w\}, \text{ and } -\xi_i \leq \xi_{w,i} \leq w \text{ on the event } \{\xi_i > w\};$$

for instance, this condition will be satisfied if $\xi_{w,i}$ is defined as $\xi_i \wedge w$ (a Winsorization of ξ_i) or as $\xi_i \mathbf{I}\{\xi_i \leq w\}$ (a truncation of ξ_i); cf. [15]; note that

$$\xi_{w,i} \leq \xi_i \wedge w \quad \text{and} \quad |\xi_{w,i}| \leq |\xi_i|. \quad (1.4)$$

Introduce next

$$S_w := \xi_{w,1} + \cdots + \xi_{w,n}.$$

Let Z stand for a standard normal r.v. Let A , possibly with subscripts, denote positive real constants depending only on the values of the corresponding parameters; more precisely, let A stand for an expression which takes positive real values depending – in a continuous manner – only on the values of the subscripts; let us allow such expressions to be different in different contexts. For any two expressions \mathcal{E}_1 and \mathcal{E}_2 , let us also write $\mathcal{E}_1 \leq_{\bullet} \mathcal{E}_2$ in place of $\mathcal{E}_1 \leq A_{\bullet} \mathcal{E}_2$, where \bullet stands for the relevant subscript(s).

Take also any $\lambda \in (0, \infty)$.

Theorem 1.1. *For all $z \in \mathbb{R}$*

$$|\mathbf{P}(S_w > z) - \mathbf{P}(Z > z)| \leq_{v,w,\lambda} \frac{\beta_v}{e^{\lambda z}}. \quad (1.5)$$

The necessary proofs will be given in Section 2.

Taking (1.5) with $\xi_{w,i} = \xi_i \mathbf{I}\{\xi_i \leq w\}$, $v = w = 1$ and $\lambda = 1/2$, and then replacing there β_1 by its upper bound μ_3 , one has, as a corollary, the result obtained by Chen and Shao [5, (6.15)].

An advantage of such bounds is that they decrease fast as $z \rightarrow \infty$. Moreover, if the right tails of (the distributions of) the r.v.'s ξ_i are light enough, then $\mathbf{P}(S_w > z)$ will differ little from $\mathbf{P}(S > z)$ for large z . This observation can be formalized in a variety of ways, some of which are presented in the following proposition.

Proposition 1.2. *For all positive p , w , c , y , and z*

$$0 \leq \mathbf{P}(S > z) - \mathbf{P}(S_w > z) \leq P_1 \wedge \cdots \wedge P_5, \quad (1.6)$$

where

$$\begin{aligned}
 P_1 &:= \mathbb{P}(\max_i \xi_i > w), \\
 P_2 &:= \mathbb{P}(\max_i \xi_i > y) + Q(z, y) \sum_i \mathbb{P}(\xi_i > w), \\
 Q(z, y) &:= \max_i \mathbb{P}(S - \xi_i > z - y, \max_{j \neq i} \xi_j \leq y), \\
 P_3 &:= \mathbb{P}(\max_i \xi_i > y) + 2Q_*(z, y) \mathbb{P}(\max_i \xi_i > w), \\
 Q_*(z, y) &:= Q(z, y) \vee \mathbb{P}(S > z, \max_j \xi_j \leq y), \\
 P_4 &:= \mathbb{P}\left(\max_i \xi_i > \frac{z}{1 + p/2}\right) + \frac{A_{p,c}}{(c+z)^p} \mathbb{P}(\max_i \xi_i > w), \\
 P_5 &:= \frac{A_{p,w,c} \mu_p}{(c+z)^p}.
 \end{aligned} \tag{1.7}$$

Upper bounds on $Q(z, y)$ and $Q_*(z, y)$ can be obtained using exponential bounds on large deviation probabilities such as ones due to Bennett [1], Hoeffding [7], and Pinelis and Utev [18]; in fact, the Bennett–Hoeffding inequality was used to obtain the bounds P_4 and P_5 defined above. One can also use bounds that are better than the corresponding exponential ones, such as ones in [14, 2, 12].

Combining (1.5) and (1.6), one immediately obtains

Corollary 1.3. *For all positive p, v, w, λ, c , and z*

$$|\mathbb{P}(S > z) - \mathbb{P}(Z > z)| \leq_{p,v,w,\lambda,c} \frac{\beta_v}{e^{\lambda z}} + (P_1 \wedge \cdots \wedge P_5). \tag{1.8}$$

An inequality similar to $|\mathbb{P}(S > z) - \mathbb{P}(Z > z)| \leq_{p,v,w,\lambda,c} \frac{\beta_v}{e^{\lambda z}} + P_5$, but with $\frac{\mu_3}{(1+z)^p}$ in place of $\frac{\beta_v}{e^{\lambda z}}$, was obtained for $p \geq 3$ by Pipiras [19], which in turn is an extension of the corresponding result by Osipov [10] for identically distributed ξ_i ’s.

In the “i.i.d.” case, when the ξ_i ’s are copies in distribution of a r.v. X/\sqrt{n} , with $\mathbb{E}X = 0$ and $\mathbb{E}X^2 = 1$, the term $\frac{\beta_v}{e^{\lambda z}}$ in (1.8) is $\leq_v \frac{\mathbb{E}|X|^3}{\sqrt{n}} e^{-\lambda z}$, which decreases fast in z ; at that, up to a constant factor, the bound P_5 is $\frac{\mathbb{E}|X|^p}{(c+z)^p n^{p/2-1}}$, which decreases fast both in z and n if p is taken to be large and $\mathbb{E}|X|^p < \infty$.

Of the other known Berry–Esseen-type bounds, the one closest to (1.5) and (1.8) in form is apparently as follows:

$$|\mathbb{P}(S > z) - \mathbb{P}(Z > z)| \leq A \sum_{i=1}^n \mathbb{E} \left(\frac{\xi_i^2}{(|z|+1)^2} \wedge \frac{|\xi_i|^3}{(|z|+1)^3} \right) \leq_v \beta_v. \tag{1.9}$$

This result was obtained in a slightly more general form by Bikelis [3, Theorem 4] (see also [11, Chapter V, Supplement 24]), and in its present form by Chen and Shao [4, Theorem 2.2].

The improvement of the existing results provided by Theorem 1.1 of this note was needed in [16] to determine the optimal zone of large deviations in which a certain nonuniform Berry–Esseen-type bound for nonlinear statistics holds.

2. Proofs

An important role in the proof of Theorem 1.1 is played by the following upper bound on the concentration probability.

Proposition 2.1. *For all real numbers a and b such that $a < b$ and all $i = 1, \dots, n$*

$$\mathbb{P}(a \leq S_w - \xi_{w,i} \leq b) \leq_{v,w,\lambda} (b - a + \beta_v) e^{-\lambda a}. \quad (2.1)$$

In the case when $\xi_{w,i} = \xi_i \mathbf{I}\{\xi_i \leq w\}$, $v = w = 1$, and $\lambda = 1/2$, an inequality similar to (2.1), with μ_3 in place of β_v , was obtained by Chen and Shao [5, Proposition 6.1].

Proof of Proposition 2.1. This proof mainly follows the lines of the proof of the mentioned Proposition 6.1 in [5]. Just as there, one can use the Bennett–Hoeffding inequality [5, (6.2)] to see that without loss of generality (w.l.o.g.) β_v is small enough, say $\beta_v \leq 0.5/v^2$; cf. the main case $\mu_3 \leq 0.17$ in the proof of Proposition 6.1 in [5]. Introduce now

$$\delta := v^3 \beta_v / 2;$$

cf. [6, Remark 2.2]. Then the condition $\beta_v \leq 0.5/v^2$ yields $\delta \in (0, v/4]$, and so, the inequality $x(x \wedge d) \geq x^2 - \frac{v^3}{4d} g(\frac{x}{v})$ for all $x \in [0, \infty)$ and $d \in (0, v/4]$ implies that

$$\sum_1^n \mathbb{E} |\xi_j| (|\xi_j| \wedge \delta) \geq 1 - \frac{v^3}{4\delta} \beta_v = \frac{1}{2}; \quad (2.2)$$

here we also used (1.1). Therefore (cf. [5, (6.8)]) and in view of the inequality $|x| \mathbf{I}\{x > w\} \leq k g(\frac{x}{v})$ for $k := \frac{v^2}{w^2} (v \vee w)$ and all $x \in \mathbb{R}$,

$$\begin{aligned} \sum_{j \neq i} \mathbb{E} |\xi_j| (|\xi_{w,j}| \wedge \delta) &\geq \sum_{j \neq i} \mathbb{E} |\xi_j| [(|\xi_j| \wedge \delta) - \delta \mathbf{I}\{\xi_j > w\}] \\ &\geq -\delta \mathbb{E} |\xi_i| - k\delta\beta_v + \sum_1^n \mathbb{E} |\xi_j| (|\xi_j| \wedge \delta) \\ &\geq -\delta - k\delta\beta_v + 0.5 = -\frac{1}{2} v^3 \beta_v (1 + k\beta_v) + 0.5 \geq 0.4, \end{aligned}$$

if β_v is assumed (w.l.o.g.) to be small enough.

The penultimate inequality in the last display follows by (2.2) and (1.1), which latter implies $\mathbb{E} |\xi_i| \leq \sqrt{\mathbb{E} \xi_i^2} \leq 1$. This trivial upper bound on $\mathbb{E} |\xi_i|$ can be improved, in an optimal way, as follows. Note that

$$u \leq \frac{2k}{3} + \frac{g(u)}{3k^2} \quad \text{for all } u \in [0, \infty) \text{ and } k \in (0, \frac{8}{9}], \quad (2.3)$$

where the function g is as defined in (1.2). This Young-type inequality can be quickly verified using the Mathematica command `Reduce[ForAll[u, u >= 0, \[CapitalDelta][k,u] >= 0] && k > 0]`, which outputs $0 < k \leq 8/9$; here the argument of the command stands for the condition $(\forall u \in [0, \infty)) \Delta(k, u) \geq 0$ & $k \in (0, \infty)$, where $\Delta(k, u) := \frac{2k}{3} + \frac{g(u)}{3k^2} - u$. Alternatively, (2.3) can be checked as follows. First, note that for all $u \in [0, \infty)$ and $k \in (0, \frac{8}{9}]$ one has $\Delta(k, u) \geq \Delta_*(u) := \Delta(k_u, u)$, where $k_u := g(u)^{1/3} \wedge \frac{8}{9} = g(u \wedge \frac{8}{9})^{1/3}$. If now $u \in [0, \frac{8}{9}]$ then $k_u = u$ and $\Delta_*(u) = 0$; if $u \in [\frac{8}{9}, 1]$ then $k_u = \frac{8}{9}$ and $\Delta_*(u) = \frac{27}{64}(u - \frac{8}{9})^2(u + \frac{16}{9}) \geq 0$; and if $u \in [1, \infty)$ then $k_u = \frac{8}{9}$ and $\Delta_*(u) = \frac{27}{64}(u - \frac{32}{27})^2 \geq 0$. This proves (2.3). Assuming now, w.l.o.g., that $\beta_v \leq (\frac{8}{9})^3$ one can use (2.3) with $k = \beta_v^{1/3} \leq \frac{8}{9}$ to see that $\mathbb{E} \frac{|\xi_i|}{v} \leq \frac{2\beta_v^{1/3}}{3} + \frac{\beta_v}{3\beta_v^{2/3}} = \beta_v^{1/3}$, whence

$$\mathbb{E} |\xi_i| \leq v\beta_v^{1/3}.$$

The latter upper bound, $v\beta_v^{1/3}$, is no greater than the bound $\mu_3^{1/3}$ on $\mathbb{E} |\xi_i|$ used in [5]. Moreover, the bound $v\beta_v^{1/3}$ is the best possible, for any given $v \in (0, \infty)$. Indeed, let $n \geq 2$, $|\xi_1| \equiv x$, and $|\xi_i| \equiv y$ for $i = 2, \dots, n$, where $n \rightarrow \infty$, $x := 1/(1 + (n-1)^{1/6})^{1/2}$, and $y := x/(n-1)^{5/12}$; then (1.1) holds, $\beta_v \leq (\frac{8}{9})^3$ eventually, and $\mathbb{E} |\xi_i| \sim v\beta_v^{1/3}$.

The other modifications that one needs to make in the proof of [5, Proposition 6.1] are rather straightforward. Thus, the proof of Proposition 2.1 is complete. \square

Proof of Theorem 1.1. This proof mainly follows the lines of the proof of the mentioned inequality [5, (6.15)]. First, in view of (1.9), w.l.o.g. $z \geq 0$. Also, just to simplify the presentation, assume, as in [5], that $v = w = 1$; the modifications needed for general v and w in $(0, \infty)$ are straightforward. One can write, as in [5, (6.16)],

$$\mathbb{P}(S_1 \leq z) - \mathbb{P}(Z \leq z) = R_1 + R_{2,1} + R_{2,2} + R_3, \quad (2.4)$$

where

$$\begin{aligned} R_1 &:= \sum_i \mathbb{E} \xi_i^2 \mathbf{I}\{\xi_i > 1\} \mathbb{E} f'_z(S_1), \\ R_{2,1} &:= \sum_i \int_{-\infty}^1 [\mathbb{P}(S_1 \leq z) - \mathbb{P}(S_1 - \xi_{1,i} + t \leq z)] \overline{K}_i(t) dt, \\ R_{2,2} &:= \sum_i \int_{-\infty}^1 [\mathbb{E} S_1 f_z(S_1) - \mathbb{E}(S_1 - \xi_{1,i} + t) f_z(S_1 - \xi_{1,i} + t)] \overline{K}_i(t) dt, \\ R_3 &:= \sum_i \mathbb{E} \xi_i \mathbf{I}\{\xi_i > 1\} \mathbb{E} f_z(S_1 - \xi_{1,i}), \end{aligned}$$

where $\overline{K}_i(t) := \mathbb{E} |\xi_{1,i}| \mathbf{I}\{|t| \leq |\xi_{1,i}|, t\xi_{1,i} > 0\}$, $f_z(s) := \Phi(z)r(s) \mathbf{I}\{s > z\} + \Phi(-z)r(-s) \mathbf{I}\{s \leq z\}$ is the Stein function, $r(s) := \Phi(-s)/\varphi(-s)$ is the Mills'

ratio, and Φ and φ are, respectively, the distribution and density functions of the standard normal distribution.

The terms R_1 and R_2 are bounded as in [5], but using the inequalities $\sum_i \mathbb{E} \xi_i \mathbf{I}\{\xi_i > 1\} \leq \sum_i \mathbb{E} \xi_i^2 \mathbf{I}\{\xi_i > 1\} \leq \beta_1$ and [5, (6.2)] with $t = \lambda$, to get

$$|R_1| + |R_3| \leq_{\lambda} \beta_1 e^{-\lambda z}. \quad (2.5)$$

Next, use Proposition 2.1 and the fact that $(z - t) \wedge (z - \xi_{1,i}) \geq z - 1$ for $t \leq 1$, to write

$$|R_{2,1}| \leq_{\lambda} e^{-\lambda z} \sum_i \mathbb{E} f_{2,1}(\xi_{1,i}, \eta_{1,i}), \quad (2.6)$$

where $\eta_{1,i}$ is an independent copy of $\xi_{1,i}$ and

$$f_{2,1}(x, y) := \int_{-\infty}^1 (1 \wedge (|x| + |t| + \beta_1)) |y| \mathbf{I}\{|t| \leq |y|\} dt. \quad (2.7)$$

In turn, one can bound $f_{2,1}(x, y)$ in two ways: for all real x and y

$$\begin{aligned} f_{2,1}(x, y) &\leq \int_{-\infty}^1 |y| \mathbf{I}\{|t| \leq |y|\} dt \leq 2y^2 \leq 2\beta_1 y^2 + 3(x^2 \vee y^2) \quad \text{and} \\ f_{2,1}(x, y) &\leq \int_{-\infty}^1 (|x| + |t| + \beta_1) |y| \mathbf{I}\{|t| \leq |y|\} dt \leq 2\beta_1 y^2 + 3(|x|^3 \vee |y|^3), \end{aligned}$$

so that $f_{2,1}(x, y) \leq 2\beta_1 y^2 + 3g(x) + 3g(y)$, where g is as in (1.2). Now (2.6), (1.1), and (1.4) yield

$$|R_{2,1}| \leq_{\lambda} e^{-\lambda z} \beta_1. \quad (2.8)$$

The term $R_{2,2}$ is bounded similarly to $R_{2,1}$. Here, instead of Proposition 2.1, one can use [5, Lemma 6.5] (with the factor $A_{\lambda} e^{-\lambda x}$ in place of $C e^{-z/2}$) and [5, (2.7)]. So, one obtains inequality (2.6) with $R_{2,1}$ and $f_{2,1}$ replaced by $R_{2,2}$ and $f_{2,2}$, respectively, where $f_{2,2}$ is obtained from $f_{2,1}$ by removing the summand β_1 from (2.7). Thus, one has (2.8) with $R_{2,2}$ in place of $R_{2,1}$. On recalling also (2.4) and (2.5), this completes the proof of Theorem 1.1. \square

Proof of Proposition 1.2. The first inequality in (1.6) is obvious. Let, for brevity,

$$\Delta_w(z) := \mathbb{P}(S > z) - \mathbb{P}(S_w > z).$$

The upper bound P_1 on $\Delta_w(z)$ is obvious.

Next, write (cf. e.g. [5, (6.13) and on] or [8, 9, 17])

$$\begin{aligned} \mathbb{P}(S > z) - \mathbb{P}(S_w > z) &\leq \mathbb{P}(S > z, \max_i \xi_i > w) \\ &\leq \mathbb{P}(\max_i \xi_i > y) + \mathbb{P}(S > z, y \geq \max_i \xi_i > w) \\ &\leq \mathbb{P}(\max_i \xi_i > y) + \sum_i \mathbb{P}(S - \xi_i > z - y, \max_{j \neq i} \xi_j \leq y) \mathbb{P}(\xi_i > w), \end{aligned}$$

which implies that $\Delta_w(z) \leq P_2$.

It is also easy to see that

$$\sum_i \mathbb{P}(\xi_i > w) \leq \frac{\mathbb{P}(\max_i \xi_i > w)}{1 - \mathbb{P}(\max_i \xi_i > w)}$$

if $\mathbb{P}(\max_i \xi_i > w) \neq 1$. So, in the case when $\mathbb{P}(\max_i \xi_i > w) \leq 1/2$ one has $\sum_i \mathbb{P}(\xi_i > w) \leq 2\mathbb{P}(\max_i \xi_i > w)$ and hence $\Delta_w(z) \leq P_2 \leq P_3$. In the other case, when $\mathbb{P}(\max_i \xi_i > w) > 1/2$, use the obvious inequalities

$$\begin{aligned} \Delta_w(z) &\leq \mathbb{P}(S > z) \leq \mathbb{P}(\max_i \xi_i > y) + \mathbb{P}(S > z, \max_j \xi_j \leq y) \\ &\leq \mathbb{P}(\max_i \xi_i > y) + Q_*(z, y) \leq P_3. \end{aligned}$$

Thus, in either case $\Delta_w(z) \leq P_3$.

Concerning the bound P_4 , choose

$$y = \frac{z}{1 + p/2}$$

in (1.7) and use the Bennett–Hoeffding inequality (see e.g. [13, Theorem 8.2]) to write

$$Q_*(z, y) \leq \left(\frac{e}{(z - y)y} \right)^{(z - y)/y} \leq_p \frac{1}{z^p} \leq_p \frac{1}{(c + z)^p}$$

in the case when $z \geq c$. So, in this case one has $\Delta_w(z) \leq P_3 \leq P_4$. If now $z \in [0, c]$ then $1 \leq_{p,c} \frac{1}{(c + z)^p}$ and hence $\Delta_w(z) \leq P_1 \leq P_4$. Thus, in either case $\Delta_w(z) \leq P_4$.

Finally, in the case when $z \geq c$, the inequality $\Delta_w(z) \leq P_5$ follows by Chebyshev's inequality from $\Delta_w(z) \leq P_4$ and (1.3). If now $z \in [0, c]$ then $\Delta_w(z) \leq P_1 \leq \frac{\mu_p}{w^p} \leq_{p,w,c} \frac{\mu_p}{(c + z)^p}$, so that in either case $\Delta_w(z) \leq P_5$. \square

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